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EXTENSION OF THE PROBLEM OF VISCOUS FLUID FLOW IN A CORNER TO THE CASE OF FLOW WITH CURVED BOUNDARIES†

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The slow flow of a viscous fluid over the solid surface which intersects another boundary surface at an angle is considered. The flow is axisymmetric and the surface contours are curved. There is no shear stress on the second surface as in the case of a free surface. The flow is investigated near the line of intersection at arbitrary angles. Formulae for the stream function and the normal stress at the boundary are obtained for short distances from the line of intersection. The leading term in the expansion corresponds to the well-known solution of the problem of flow in a corner. The second term of the expansion of the normal stress at the free surface takes into account the curvature of the flow boundaries. The axial symmetry of the flow and the curvature of the boundary contours lead to a logarithmic singularity in the normal stress.

Investigations of the slow motions of a viscous fluid in a corner were initiated by Rayleigh [1] who considered the case of stationary sides of the corner. Taylor [2] studied the problem of the flow in the corner formed by two plane solid boundaries, one of which moves. Unlike Taylor's problem, in Moffat's problem [3] there is no shear stress on one side of the corner as in the case of a free fluid surface.

We will consider a similar problem when the intersecting surfaces are not plane. In this extension it seems natural to refer to the analysis of the flow at short distances from the line of intersection (the contact line). In this case, it is possible to extend the solution of the problem of the flow in a corner to a wider class of flows as an approximate solution. The validity of this extension can be provided by the second term of the asymptotic form, which has not been considered so far even in special cases.

The problem is of interest in the context of fluid mechanics of the wetting of solids by viscous fluids. The contact angle can be formed due to the motion of a fluid with a free boundary in a small vicinity of the moving contact line. The effect of the dynamic contact angle is based on a slow variation of the slope of the boundary with distance according to a non-linear asymptotic form [4–6]. This asymptotic form is valid in an intermediate range of small scales near the contact line, where the distances are bounded below by the minimum (microscopic) scale. The microscale may be due to the dynamics of a hyperfine percursion film, which moves ahead of the spread of the fluid with a contact angle, as has been established in the asymptotic theory [4.5] and was confirmed by de Gennes [7].

If a small range of the above-mentioned non-linear asymptotic form exists, another effect is possible: at large scales the dynamics of the fluid can only slightly affect the shape of the free surface. This leads to a model of a quasi-static surface at large scales [4], which has the following meaning.

At a liquid–gas interface S, Laplace condition for the mean curvature H holds

$$2\sigma H = P_n + p_0$$

where P_n is the normal stress in the fluid, p_0 is the gas pressure, and σ is the surface tension. In the large-scale region the stress P_n differs slightly from the static value, that is, from a constant, if there are no body forces. Hence, the shape of the interface is approximately defined by the equation of capillary statics, and in this case it may depend on time. For the static surface S, one can find the stress P_n from the viscous flow problem. The surface shape can then be refined, i.e. one can obtain a slightly perturbed shape from the boundary condition with varying stress P_n. Hence, the perturbed shape of the surface S will be quasi-static. This model of the dynamics of a spreading fluid

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with a free surface was pointed out in [4] for improving the shape of a spreading drop, close to a spherical segment of varying radius and arbitrary contact angle. The model includes the case of small angles as a special one.

The model of the spreading of a drop [4] was completed with formulae [5] for the general parameters of the spreading model for the case when there is a moving precursion film. The model [4] of the quasi-stationary state of the drop surface in the central domain has been used in many papers for the case of small angles, and this is reflected in the survey by de Gennes [7].

Thus, in the context of the fluid dynamics of the wetting of a solid surface, the problem of viscous fluid flow over a solid surface with a static form of free boundary is of some intersect.

1. FORMULATION OF THE PROBLEM AND THE METHOD OF SOLUTION

Consider the axisymmetric flow of a viscous fluid over the solid surface at low Reynolds numbers. The free boundary S intersects of the solid surface S_{sol} along a moving contact line. The problem is considered in a plane passing through the axis of symmetry of the system; on this plane the boundary surfaces are represented by contours (generatrices). Suppose the solid body is stationary. On its surface the fluid velocity vanishes

$$\boldsymbol{v} = 0 \quad \text{at} \quad S_{\text{sol}} \tag{1.1}$$

On the free surface S the normal velocity of the fluid is equal to the normal velocity of the surface w.

$$\boldsymbol{v} \cdot \mathbf{n} = w \tag{1.2}$$

where $\mathbf{n}$ is the unit vector of the outward normal to the fluid surface S.

If the contour S moves as a rigid body and does not rotate, condition (1.2) takes the form

$$\boldsymbol{v} \cdot \mathbf{n} = v_0 n_1, \quad \text{where} \quad n_1 = \mathbf{n} \cdot \mathbf{e}_1 \tag{1.3}$$

where v_0 is the velocity of the contact line and $\mathbf{e}_1$ is the unit vector of the tangent directed to the dry part of the solid surface.

In the general case, we specify the velocity w in the limit of a short distance r from the contact line

$$\boldsymbol{v} \cdot \mathbf{n} = v_0 n_1 + \omega_F r + \dots, \quad r \to 0 \tag{1.4}$$

Here ω_F is the angular velocity of rotation of the unit vector τ_F of the tangent to the contour S at a point of the contact line.

The shear stress P_τ vanished on the contour S:

$$P_\tau = \boldsymbol{\tau} \cdot \mathbf{P} \cdot \mathbf{n} = 0 \tag{1.5}$$

where $\mathbf{P}$ is the stress tensor in the fluid and τ is the unit vector of the tangent to the contour S.

We will introduce a cylindrical system of coordinates x and z, where x is the distance from the axis of symmetry and z is the coordinate along this axis.

We will write the equations for the stream function and the pressure

$$\hat{E}^2\psi = 0, \quad \hat{E} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{x}\frac{\partial}{\partial x} \quad \left(v_x = -\frac{1}{x}\frac{\partial \psi}{\partial z}, v_z = \frac{1}{x}\frac{\partial \psi}{\partial x}\right) \tag{1.6}$$

$$\frac{1}{\mu}\frac{\partial p}{\partial x} = -\frac{1}{x}\frac{\partial}{\partial z}\hat{E}\psi, \quad \frac{1}{\mu}\frac{\partial p}{\partial z} = \frac{1}{x}\frac{\partial}{\partial x}\hat{E}\psi \tag{1.7}$$

If the distance from the contact line is short, a local description of the geometry of the boundaries is sufficient. We will assume that the boundary surfaces are smooth, and hence the coordinates of their points can be defined by a few terms of the Taylor series expansion.

We will denote the angle between the intersecting boundaries by α.

We will also consider the second angle γ, which the tangent vector $\mathbf{e}_1$ makes with the radial axis x (see Fig. 1). We will assume that $\gamma > 0$ if the outward normal to the solid surface is directed towards the axis of symmetry, and $\gamma < 0$ if it is directed from the axis of symmetry.

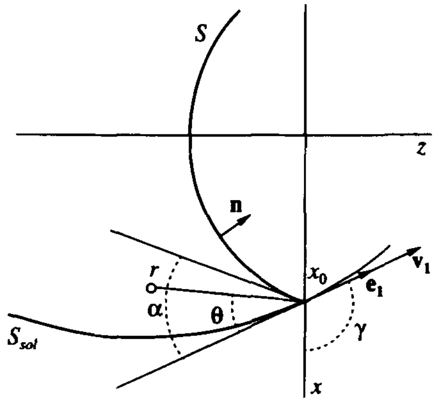

Fig. 1

We will denote the curvatures of the contours of the surfaces S and $S_{\rm sol}$ on the contact line by k_F and k_S respectively. The point of the contact line has the coordinates $x = x_0$ and $z = 0$, where x_0 is the radius of the contact line.

We will introduce a polar system of coordinates r and θ with origin at the point of the contact line when the limit $\theta = 0$ is in contact with the solid surface and is directed towards its wetted part (see Fig. 1). Near the contact line the contour $S_{\rm sol}$ is close to the tangent:

$$\theta = \frac{1}{2}k_S r + O(r^2), \quad r \to 0 \tag{1.8}$$

For small r the free boundary contour S is described by the equation

$$\theta - \alpha = \frac{1}{2}k_F r + O(r^2), \quad r \to 0 \tag{1.9}$$

Hence, the geometry of the boundaries is specified by five parameters, namely, the angles α and γ, the radius x_0 of the contact line or its curvature x_0^{-1}, and the contour curvatures k_F and k_S.

On the solid surface the stream function vanished, $\psi = 0$, and hence $\psi \to 0$ as $r \to 0$, due to the absence of a singularity on the contact line.

In the special case of plane boundaries when the contact line is a straight line, the solution is known [3]. We will denote the corresponding stream function and the normal stress at the free surface by ψ_0 and P_{n0}. In the general case, the function ψ for small values of r can be represented by the expansion

$$\psi = \psi_0 + \psi_1 + \dots \tag{1.10}$$

The second term ψ_1 is a small correction, $\psi_1 \ll \psi_0$ as $r \to 0$. The first term has the well-known from $\psi_0 = rf(\theta)$ and the second term can be sought in the form $\psi_1 = r^2 f_1(\theta)$.

Using the linearity of the problem, we can conclude that the perturbation of the stream function ψ_1 is a linear function of the curvatures k_0, k_F and k_S. Hence, the problem of finding ψ_1 can be split into three different problems with the following conditions:

(1) The curvature of the contact line is non-zero, $k_0 \neq 0$, whereas the two other curvatures vanish.

(2) The curvature of the free boundary contour is non-zero, $k_F \neq 0$, and $k_0 = k_S = 0$.

(3) The curvature of the solid wall contour is non-zero, $k_S \neq 0$, and $k_0 = k_F = 0$

Below we present the solutions of these three problems sequentially. When solving the third problem we shall also analyse the possible influence of the non-stationary state of the boundaries.

2. PROBLEM 1: FLOW WITH AXIAL SYMMETRY

Assuming $k_F = 0$ and $k_S = 0$, we take into account that $k_0 \neq 0$. We will assume that the tangent to the free boundary contour at the point of the contact line does not rotate, that is, $\omega_F = 0$.

From relations (1.3) and (1.9) at the free boundary we have

$$v_\theta(r, \theta) = v_0 \sin\alpha + O(r^2) \text{ for } \theta = \alpha + O(r^2) \tag{2.1}$$

Condition (1.5) for the shear stress to vanish gives

$$P_{r\theta}(r,\theta) = O(r) \text{ for } \theta = \alpha + O(r^2) \tag{2.2}$$

where

$$\frac{1}{\mu}P_{r\theta} = \frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \tag{2.3}$$

In polar coordinates the velocity components are expressed in terms of the stream function as follows:

$$v_r = \frac{1}{rx}\frac{\partial \psi}{\partial \theta}, \quad v_\theta = -\frac{1}{x}\frac{\partial \psi}{\partial r} \tag{2.4}$$

Substituting expansion (1.10) into Eq. (1.6), in the main approximation we obtain the biharmonic equation

$$\Delta_0^2\psi_0 = 0, \quad \Delta_0 = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{2.5}$$

In the main approximation, boundary conditions (1.1) on the solid wall give

$$\psi_0 = 0, \quad \frac{\partial \psi_0}{\partial \theta} = 0 \text{ when } \theta = 0 \tag{2.6}$$

Here we also add boundary conditions (2.1) and (2.2), written in the main approximation as $r \to 0$. We will write the well-known solution of the problem for Eq. (2.5) with these conditions in the form

$$\psi_0 = rx_0 f_0(\theta) \tag{2.7}$$

$$f_0 = v_0 Q[\sin\theta\cos\alpha - \theta\cos\theta - \alpha], \quad Q = \sin\alpha/(\alpha - \sin\alpha\cos\alpha) \tag{2.8}$$

The solutions of the problem with homogeneous boundary conditions on the sides of the corner and with the condition that they decrease as $r \to 0$, decrease more rapidly than ψ_0 for any $\alpha < \pi$. If $\alpha = \pi$, the solution degenerates: $\psi_0 \equiv 0$.

The equation for the perturbation ψ_1 follows from relations (1.6), (1.10) and (2.5)

$$\Delta_0^2\psi_1 = \frac{2}{x_0}\frac{\partial}{\partial x}\Delta_0\psi_0$$

Using expressions (2.7) and (2.8), we obtain

$$\Delta_0^2\psi_1 = \frac{4}{r^2}v_0 Q(\alpha)\sin(2\theta - \alpha - \gamma) \tag{2.9}$$

We will now derive the boundary conditions for ψ_1. The conditions on the solid surface give

$$\psi_1(r,0) = 0, \quad \frac{\partial \psi_1}{\partial \theta}(r,0) = 0 \tag{2.10}$$

Using the equality

$$x - x_0 = -r\cos(\theta - \gamma)$$

from relations (2.1) and (2.4) we find the condition on the free boundary

$$\frac{\partial \psi_1}{\partial r}(r,\alpha) = v_0 r\sin\alpha\cos(\alpha - \gamma) \tag{2.11}$$

From relations (2.1)–(2.4) it follows that

$$\frac{\partial^2\psi}{\partial\theta^2} = \frac{1}{x}\frac{\partial x}{\partial\theta}\frac{\partial\psi}{\partial\theta} + xrv_0\sin\alpha + \dots \quad \text{when} \quad \theta = \alpha$$

Substituting expansion (1.10) here, we obtain

$$\frac{1}{r^2}\frac{\partial^2\psi_1}{\partial\theta^2}(r,\alpha) = -v_0\sin\alpha\cos(\alpha-\gamma) - v_0 Q\sin^2\alpha\sin(\alpha-\gamma) \tag{2.12}$$

As a result we have boundary conditions (2.10), (2.11) and (2.12) on the sides of the corner $\theta = 0$ and $\theta = \alpha$. The particular solution of the problem for Eq. (2.9) with these conditions has the form

$$\begin{aligned} \psi_1 &= r^2 f_1(\theta); \\ f_1 &= \frac{1}{4}v_0 Q\{-\cos\alpha\sin(2\theta-\gamma) - \cos\alpha\sin\gamma + \theta\cos(2\theta-\alpha-\gamma) + \theta\cos(\alpha-\gamma)\} \end{aligned} \tag{2.13}$$

We will determine the normal stress on the free boundary S near the contact line. In the limit as $r \to 0$ and taking into account relations (1.9), (2.7) and (2.13), the normal stress can be written in the form

$$P_n|_S = \mathbf{n}\cdot\mathbf{P}\cdot\mathbf{n} = P_{\theta\theta}(r,\theta) + \dots \quad \text{when} \quad \theta = \alpha + O(r^2)$$

$$P_{\theta\theta} = -p + 2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r}\right) = -p + O(1)$$

Hence

$$P_n|_S = -p(r,\alpha) + O(1) \quad \text{when} \quad r \to 0 \tag{2.14}$$

Now we rewrite Eq. (1.7) in polar coordinates

$$\frac{1}{\mu r}\frac{\partial p}{\partial\theta} = -\frac{1}{x}\frac{\partial}{\partial r}\left(\Delta_0\psi - \frac{1}{x}\nabla_x\psi\right), \quad \frac{1}{\mu}\frac{\partial p}{\partial r} = \frac{1}{xr}\frac{\partial}{\partial\theta}\left(\Delta_0\psi - \frac{1}{x}\nabla_x\psi\right) \tag{2.15}$$

Using the expansion

$$\frac{1}{x} = \frac{1}{x_0} - \frac{x-x_0}{x_0^2} + \dots$$

and the identities

$$\frac{\partial}{\partial\theta}\Delta_0\psi_0 = \frac{2x_0}{r}v_0 Q\cos(\theta-\alpha)$$

$$\frac{\partial}{\partial\theta}\Delta_0\psi_1 = v_0 Q\cos(\alpha-\gamma) - 2v_0 Q\cos(2\theta-\alpha-\gamma)$$

$$\frac{1}{x_0}\frac{\partial}{\partial\theta}\nabla_x\psi_0 = 2v_0 Q\sin(\theta-\gamma)\sin(\theta-\alpha)$$

from relations (2.15) we find

$$\frac{1}{\mu}\frac{\partial p}{\partial r} = \frac{2v_0}{r^2}Q\cos(\theta-\alpha) + \frac{v_0}{x_0 r}Q\cos(\alpha-\gamma) + \dots$$

Integrating this identity, we obtain

$$\frac{1}{\mu}p = -\frac{2v_0}{r}Q\cos(\theta-\alpha) + \frac{v_0}{x_0}Q\cos(\alpha-\gamma)\ln r + O(1) \tag{2.16}$$

From relations (2.14) and (2.16) it follows that as $r \to 0$ the normal stress on the boundary S is given by the formula

$$P_n|_S = \frac{2}{r}\mu v_0 Q + P_{n1}, \quad P_{n1} = -\frac{\mu v_0}{x_0}Q\cos(\alpha-\gamma)\ln r + O(1) \tag{2.17}$$

The first term in the expansion of P_n (2.17) corresponds to the plane problem of the flow in a corner.The second term P_{n1} is related to the axial symmetry of the flow, and it is essential that this term is infinite at the point $r = 0$. The term P_{n1} depends on the angle γ between the radial axis and the unit vector $\mathbf{e}_1$ of the velocity of contact line (see Fig. 1).

We will consider various values of this angle.

If the fluid flows in a circular tube, $\gamma = \pi/2$.

In the fluid flows over the surface of a solid rod, $\gamma = -\pi/2$.

Suppose the fluid flows over a plane surface. Assume that its wetted part is a circle; then $\gamma = 0$, which corresponds, in particular, to a liquid drop on a surface. If the dry part of the plane solid surface is a circle, we have $\gamma = \mp\pi$; this relates to the case of a gas bubble in contact with a wall.

A remark on the role of the solutions of the homogeneous problem of the flow in a corner. As can be seen, the analysis of the flow in the limit of short distances from the contact line leads to the problem of the flow in a corner for the inhomogeneous equation in ψ_1 (1.10) with inhomogeneous conditions on the sides of the corner. It is possible to transform the solution of this problem by adding the arbitrary solution $\tilde{\psi}$ of the biharmonic equation with homogeneous boundary conditions on the sides of the corner, where $\tilde{\psi} \to 0$ as $r \to 0$. In this case the asymptotic solution $\psi = \psi_0 + \psi_1 + \ldots$ is unchanged if all the solutions of the homogeneous problem decrease more rapidly than r^2 as $r \to 0$, $\tilde{\psi} = o(r^2)$. The solutions of homogeneous problem then decrease more rapidly than ψ_1 and do not contribute to the asymptotic solution. In the opposite case the contribution from the solution of the homogeneous problem is of greater importance than ψ_1.

For solutions of the form $\tilde{\psi} = r^{m+1}\psi(\theta)$ one obtains the well-known equation [3]

$$\sin 2m\alpha - m\sin 2\alpha = 0; \quad m \neq 1, \quad \mathrm{Re}\, m > -1$$

A root with real part $\mathrm{Re}\, m_1 < 1$ exists when $\alpha > \alpha_* = 128.7°$, and when $\alpha < \alpha_*$ for all roots one has $\mathrm{Re}\, m_1 > 1$. Hence, the contribution of $\tilde{\psi}$ to the asymptotic solution is of greater importance than the contribution of ψ_1, if the angle α is greater than the critical value α_*. If $\alpha < \alpha_*$, any solution of the homogeneous problem $\tilde{\psi}$ decreases more rapidly than ψ_1 as $r \to 0$ and can only contribute to the residual term in expansion (1.10).

Consequently, the asymptotic expressions for the stream function and the normal stress considered turn out to be valid when the angle of contact is less than the critical angle: $\alpha < \alpha_* = 128.7°$.

3. PROBLEM 2: FLOW WITH A CURVED CONTOUR OF THE FREE BOUNDARY

We will assume that the curvature of the contour of the free boundary k_F si non-zero and $k_0 = k_S = 0$. In this case it is sufficient to consider the plane problem. We will use a stream function, which differs from ψ in relation (2.4) by the factor x_0

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \quad v_\theta = -\frac{\partial\psi}{\partial r} \tag{3.1}$$

In this case the first term in the expansion for ψ (1.10) has the form

$$\psi_0 = rf_0(\theta) \tag{3.2}$$

The function $f_0(\theta)$ is given by expression (2.8).

Consider the unit vectors of the normal $\mathbf{n}$ and of the tangent τ to the free boundary. With an accuracy of $O(r^2)$ we have

$$n_r = -\delta, \quad \tau_\theta = \delta; \quad \delta = k_F r/2 \tag{3.3}$$

Using expression (3.3) we find the component of the normal n_1 along the unit velocity vector $\mathbf{e}_1$

$$n_1 = \sin\alpha + 2\delta(r)\cos\alpha + \ldots \tag{3.4}$$

Condition (1.5) for the shear stress to vanish gives

$$P_\tau = P_{r\theta} + (P_{\theta\theta} - P_{rr})\delta + \ldots = 0 \tag{3.5}$$

The component $P_{r\theta}$ on the surface S can be represented by the expansion

$$P_{r\theta}(r, \alpha + \delta) = P_{r\theta}(r, \alpha) + \frac{\partial P_{r\theta}}{\partial \theta}(r, \alpha)\delta + \ldots, \quad r \to 0 \tag{3.6}$$

From relations (1.10), (2.8) and (3.2) we find

$$P_{rr} - P_{\theta\theta} = O(r), \quad \frac{1}{\mu}\frac{\partial P_{r\theta}}{\partial \theta}(r, \theta) = \frac{2}{r}Qv_0 + O(1)$$

Using these formulae in relations (3.5) and (3.6) and taking into account equality (2.3), we obtain

$$\frac{1}{r^2}\frac{\partial^2 \psi_1}{\partial \theta^2} - \frac{\partial^2 \psi_1}{\partial r^2} + \frac{1}{r}\frac{\partial \psi_1}{\partial r} = -v_0 k_F Q \text{ when } \theta = \alpha \tag{3.7}$$

We will write down the condition for the normal velocity of the fluid on the surface S

$$v_\theta n_\theta + v_r n_r = v_0 n_1 + O(r^2)$$

Representing v_θ by two terms of the Taylor series expansion and taking into account expressions (3.2)–(3.4), we obtain

$$\frac{\partial \psi_1}{\partial r}(r, \alpha) = -v_0 k_F r(\cos\alpha - Q\sin^2\alpha) \tag{3.8}$$

The conditions on the solid surface S_{sol} can be transferred onto the tangent line $\theta = 0$ as $r \to 0$, since the curvature $k_S = 0$. We have

$$\psi(r, 0) = 0, \quad \frac{\partial \psi}{\partial \theta}(r, 0) = 0 \text{ when } r \to 0 \tag{3.9}$$

Consequently we arrive at the problem for the biharmonic equation with conditions (3.7)–(3.9) given on the sides of the corner $\theta = 0$ and $\theta = \alpha$. Its particular solution is

$$\psi_1 = r^2\Phi(\theta), \quad \Phi = \frac{1}{4}v_0 k_F Q[1 - \cos 2\theta - \operatorname{ctg}\alpha(2\theta - \sin 2\theta)] \tag{3.10}$$

The fluid pressure is given by the two-term expression $p = p_0 + p_1$, where the term p_0 corresponds to ψ_0, and p_1 corresponds to ψ_1. From the equation for the pressure

$$\frac{1}{\mu}\frac{\partial p}{\partial r} = \frac{\partial}{\partial \theta}\Delta\psi \tag{3.11}$$

and solution (3.10), we obtain

$$p_1 = -2\mu v_0 k_F Q \operatorname{ctg}\alpha \ln r + O(1) \tag{3.12}$$

Substituting expression (3.12) into relation (2.14), we obtain the contribution ψ_1 to the normal stress P_n on the surface S

$$P_{n1} = 2\mu v_0 k_F \frac{\cos\alpha}{\alpha - \sin\alpha\cos\alpha}\ln r + O(1) \text{ when } r \to 0 \tag{3.13}$$

The first term of the expansion of the normal stress $P_n = P_{n0} + P_{n1}$ is represented by the second formula (2.17). The second term of the expansion of the normal stress, given by expression (3.13), is infinite on the contact line if $\alpha \neq \pi/2$. Like the quantity P_{n1} (2.17), this term behaves like $\ln r$.

4. PROBLEM 3: FLOW ALONG A CURVED WALL

We will consider the plane problem when the wall curvature k_S is non-zero, but $k_F = 0$. The stream function is presented by expansion (1.10); formulae (2.8) and (3.2) specify the quantity ψ_0.

Using the boundary conditions on the surface S_{sol}, expanding the quantities ψ and v_r in Taylor series at the point $\theta = 0$, and taking into account expansion (1.10) and the expression for ψ_0, we obtain

$$\psi_1(r, 0) = 0, \quad \frac{\partial\psi_1}{\partial\theta}(r, 0) = r^2 k_S v_0 Q \sin\alpha \tag{4.1}$$

For small r the boundary conditions on the free surface can be transferred to the tangent $\theta = \alpha$, since the curvature $k_F = 0$. From relations (1.4), (1.5) and (1.10) we have

$$\frac{\partial\psi_1}{\partial r}(r, \alpha) = -\omega_F r \tag{4.2}$$

$$\frac{1}{r^2}\frac{\partial^2\psi_1}{\partial\theta^2} - \frac{\partial^2\psi_1}{\partial r^2} + \frac{1}{r}\frac{\partial\psi_1}{\partial r} = 0 \text{ when } \theta = \alpha \tag{4.3}$$

We will write the solution of the problem for the biharmonic equation in a corner with conditions (4.1)–(4.3)

$$\psi_1 = r^2\Phi(\theta)$$

$$\Phi(\theta) = c\theta + \frac{2c\alpha + \omega_F}{2D(\alpha)}\{\mathrm{tg}2\alpha(1 - \cos 2\theta) - 2\theta + \sin 2\theta\} \tag{4.4}$$

$$c = k_S v_0 Q \sin\alpha, \quad D = 2\alpha - \mathrm{tg}2\alpha$$

Substituting expression(4.4) into (3.11), we find the pressure p_1. We have

$$\frac{1}{\mu}p_1 = -\frac{4c\,\mathrm{tg}2\alpha}{D}\ln r - \frac{4\omega_F}{D}\ln r + O(1)$$

From expression (2.14) we obtain the second term of the expansion of the normal stress

$$\frac{1}{\mu}P_{n1} = 4k_S v_0 \frac{Q\sin\alpha\,\mathrm{tg}2\alpha}{D}\ln r + \frac{4\omega_F}{D}\ln r + O(1) \tag{4.5}$$

Here k_S is the curvature of the rigid wall. The second term takes into account a non-stationary state of the surface S.

Solution (4.5) holds if $\alpha \neq \alpha_*$, where α_* is the root of the equation $D(\alpha) = 0$. This quantity is identical with the critical angle α_* discussed in Section 2.

We will consider the tangent to the contour S_{sol} at the point of the contact line and direct the unit vector τ_S toward the wetted part of the wall. The wall curvature k_S produces a change of the vector τ_S when the contact line moves. The vector τ_S rotates with the angular velocity

$$\omega_S = -k_S v_0 \tag{4.6}$$

The unit vector τ_F of the tangent to the contour of the free boundary rotates with angular velocity ω_F. The angle between τ_F and τ_S is equal to α. Consequently

$$\alpha^{\cdot} = \omega_F - \omega_S \tag{4.7}$$

Replacing ω_F in equality (4.5) using relations (4.6) and (4.7), we arrive at the equation

$$\frac{1}{\mu}P_{n1} = \left\{-\frac{2k_S v_0}{\alpha - \sin\alpha\cos\alpha} + \frac{4\alpha^{\cdot}}{2\alpha - \operatorname{tg}2\alpha}\right\}\ln r + O(1) \tag{4.8}$$

which gives the contributions to the expansion of the normal stress on the free surface from the curvature of the contour of the solid body k_S and from a non-stationary state of the boundaries.

5. THE ASYMPTOTIC FORMULA FOR THE NORMAL STRESS ON THE FREE SURFACE

Summing the contributions to the normal stress on the surface S, defined by relations (2.17), (3.13) and (4.8), in the limit as $r \to 0$, we obtain the asymptotic expression for the normal stress on free boundary in the form

$$\frac{1}{\mu}P_n = \frac{2v_0}{r}\frac{\sin\alpha}{\alpha - \sin\alpha\cos\alpha} +$$

$$+\left\{v_0\frac{2k_F\cos\alpha - 2k_S - k_0\sin\alpha\cos(\alpha-\gamma)}{\alpha - \sin\alpha\cos\alpha} + \frac{4\alpha^{\cdot}}{2\alpha - \operatorname{tg}2\alpha}\right\}\ln r + O(1) \tag{5.1}$$

where v_0 is the velocity of the contact line α is the contact angle γ is the angle related to the axial symmetry of the flow (see the figure) and k_0, k_F and k_S are the curvatures of the contact line, the contour of the free surface and the contour of the solid body respectively. The contact angle can take values in the range $(0, \alpha_*)$, where $\alpha^* \approx 128.7°$.

Formula (5.1) is true for short distances form the contact line; its novelty consists in the occurrence of a logarithmic term due to bending of the flow boundaries. This term is essential, since it is infinite on the contact line.

In the special case of plane boundaries and a fixed contact angle expression (5.1) is identical with the well-known solution [3].

Note that the expression for the normal stress (5.1) has a general meaning since it holds for arbitrary axisymmetric Stokes flow with intersecting boundaries.

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